the groups consisting of shots 1–10 and 31–40. The latter group was chosen in the range where performance was essentially constant.

Molybdenum and stainless-steel electrode materials were selected for the erosion investigation, since their performance represented the extremes for materials previously considered. Molybdenum captured on the filter paper was prepared by destroying the paper with nitric and perchloric acids and evaporating the resulting solution to dryness. The salts were then dissolved in dilute perchloric acid. One-fifth of the sample solution was reserved for the determination of silver, the propellant material, whereas the remainder of the solution was used for the photometric determination of the mass of molybdenum.

The low concentrations of the constituent elements (iron, chromium, and nickel) of Stainless Steel No. 304 necessitated the development of a new method[2] for the mass determination. The sample was prepared with nitric and perchloric acid as in the case of molybdenum. The silver present was then precipitated as the chloride and removed by filtration. The sample was then deposited as a spot on chromatographic paper using the ring oven technique. A series of standard spots was prepared by using a standard solution of National Bureau of Standards (NBS) Steel 101E, which corresponds to Stainless Steel No. 304. The standard spots were used to provide the calibration curve for the x-ray fluorescence method, and the amounts of iron, chromium, and nickel in the unknown were determined after measuring their x-ray fluorescence intensity.

Table 1 Summary of electrode erosion measurements

Sample designation	Molybdenum electrodes, μg Mo	Stainless-steel electrodes, μg Fe, Ni, and Cr
Shots 1–10	61.5	16.8
Shots 31–40	25.4	5.1

The portions of the various samples reserved for determination of silver were evaporated to dryness. The mass of silver present was determined by a novel polarographic method developed by Cave and Hume.[3] The silver on the stainless-steel electrodes was removed with dilute nitric acid, and the mass determined polarographically. The mass of silver on the molybdenum electrodes could not be determined because of unexplained difficulty with the polarographic method.

The results of the measurements are summarized in Table 1. A typical example of results obtained with molybdenum electrodes during the first ten shots indicates eroded mass equal to 5.2% of the initial propellant mass. Discharge conditioning of the electrodes results in further reduction of the erosion, which approaches a value equal to 2% of the initial propellant mass. Erosion is still further reduced in the case of the stainless-steel electrodes, amounting to only 0.4% of the propellant mass with conditioned electrodes, although the over-all performance of this material was considerably below that of molybdenum from the standpoint of momentum production. The inferior performance of the stainless-steel electrodes was probably due to their relatively high resistivity.

A check on the validity of the present measurements was provided by measuring the silver propellant, the initial mass of which was known. Significantly, over 97% of this mass was accounted for by the measurement techniques. A further check in the case of stainless-steel electrodes was provided by comparing the percentages of constituent elements of the sample to the specification for AISI (American Iron and Steel Institute) type 304 stainless steel. These percentages were found to fall within the specified limits. The standard deviation was 10% for the stainless-steel analysis and 2% for the molybdenum determination. On these bases the measurements are accepted with a high degree of confidence.

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Heating of the Cavity Inside a Spherical Shell Satellite

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Introduction

THE satellite considered herein consists of a conducting spherical shell containing inside an ideal gas. It is the purpose of this paper to determine the temperature distribution both in the spherical shell and in the cavity under the boundary condition of a given heat flux applied to the outer surface. This heat flux $Q_1(\theta, t)$ may vary both with time and spherical coordinate θ. Two possible boundary conditions at the inner surface of the shell are considered: 1) the amount of heat radiated from the inner surface to the enclosed gas is proportional to the temperature gradient, and 2) the temperature at the inner surface, and also of the enclosed gas, is a prescribed function $G(\theta, t)$ of t and θ.

An asymptotic solution valid for small time has been obtained for the first problem. The solution to the second problem converges more rapidly than Warren's solution particularly near the inner boundary and for small times.

No account has been taken of the temperature variation of the thermal parameters of the material of the shell. The solutions given hold for both an aerodynamic or radiation heat flux, $Q(\theta, t)$ (with ϕ-wise symmetry). This work can be readily extended to the case with ϕ-wise heat flux variations.

Radiation Law at the Inner Surface

Boundary conditions

We assume that the heat flux applied to the outer surface of the shell, which we denote by $Q_1(\theta, t)$ heat units per unit time per unit surface area, can be expanded as a Fourier-Legendre series:

$$\frac{Q_1(\theta, t)}{K} = Q(\theta, t) = \sum_{n=0}^{\infty} q_n(t)P_n(\cos\theta) \text{ for } t > 0 \quad (1)$$

This series may be finite (if $q_n \equiv 0$ for $n \geqslant N_0$) or infinite; $P_n(\cos\theta)$ is the standard Legendre polynomial of degree n and argument $\cos\theta(= \mu)$, and K is the thermal conductivity. The boundary condition on the outer surface $r = a$ then becomes

$$\left[\frac{\partial T}{\partial r}\right]_{r=a} = Q(\theta, t) = \sum_{n=0}^{\infty} q_n(t)P_n(\mu) \quad (2)$$

where $T(r, \theta, t)$ denotes the temperature in spherical polar coordinates (r, θ, ϕ) whose origin coincides with the center of the spherical shell. At the inner surface $r = b(< a)$, we assume that the radiation condition can be expressed in the Newtonian form

$$\left[\frac{\partial T}{\partial r}\right]_{r=b} = H(T - T_i), \quad (3)$$

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where $T_i = T_i(t)$ is the uniform temperature of the gas enclosed by the shell and H is a constant. The heat balance equation for the enclosed gas in the region $0 \leqslant r < b$ is assumed to be

$$\left[\frac{\partial T}{\partial r}\right]_{r=b} = S\left(\frac{\partial T_i}{\partial t}\right) \tag{4}$$

where S is a constant ($\alpha b \rho c_v/3K$ in a usual notation).

We shall also assume that the initial conditions are

$$T(r, \theta, 0) = 0 = T_i(0) \text{ at } t = 0 \tag{5}$$

Solution for the temperature

In spherical polar coordinates (r, θ, ϕ), the heat conduction equation with ϕ-wise symmetry is

$$\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial T}{\partial \theta}\right) = \frac{1}{k}\frac{\partial T}{\partial t} \tag{6}$$

where k is the thermal diffusivity. Let $\bar{T}$, $\bar{q}_n$ be the Laplace transforms of T, q_n, then

$$\bar{T} = \bar{T}(r, \theta, p) = \int_0^\infty e^{-pt} T(r, \theta, t) dt = \mathfrak{L}\{T\} \tag{7}$$

and the general solution for $\bar{T}$ from (6) and (7) is

$$\bar{T} = \sum_{n=0}^{\infty} \{A_n i_n(qr) + B_n m_n(qr)\} P_n(\cos\theta) \tag{8}$$

where $i_n(qr)$ and $m_n(qr)$ are spherical modified Bessel functions satisfying the equation

$$r^2 d^2y/dr^2 + 2r dy/dr - \{n(n+1) + q^2 r^2\} y = 0 \tag{9}$$

Note that

$$i_n(qr) = \frac{I_{n+1/2}(qr)}{(qr)^{1/2}} \qquad m_n(qr) = \frac{K_{n+1/2}(qr)}{(qr)^{1/2}} \tag{10}$$

where I_ν and K_ν are the modified Bessel functions of the first and second kinds[3] of order ν and also $p = q^2 k$.

The transform of boundary condition (2) is satisfied if

$$\bar{q}_n(p) = q\{A_n i_n'(qa) + B_n m_n'(qa)\} \tag{11}$$

where a bar denotes differentiation with respect to, the complete argument, viz.,

$$i_n'(qa) = \frac{d\{i_n(qa)\}}{d(qa)} \tag{12}$$

Boundary conditions (3–5) lead to

$$\mathfrak{L}\{T_i\} = \bar{T}_i = \frac{H\{\bar{T}\}_{r=b}}{H + Sp} \tag{13}$$

and, in terms of $\bar{T}$ alone, the boundary condition at the inner surface becomes

$$\frac{\partial \bar{T}}{\partial r} = \frac{HSp\bar{T}}{H + Sp} \qquad \text{for } r = b \tag{14}$$

This boundary condition (14), together with (11), is used to determine the values of A_n and B_n and hence the solution, i.e.,

$$\bar{T}(r, \theta, p) = \sum_{n=0}^{\infty} P_n(\mu) \frac{\bar{q}_n}{q}\left\{\frac{L_n(q, r, b) + \Delta M_n(q, b, r)}{N_n(q, a, b) + \Delta L_n(q, b, a)}\right\} \tag{15}$$

where

$$L_n(q, r, b) = i_n(qr) m_n'(qb) - m_n(qr) i_n'(qb) \tag{16}$$

$$M_n(q, b, r) = i_n(qb) m_n(qr) - i_n(qr(m_n(qb) \tag{17}$$

$$N_n(q, a, b) = i_n(qa) m_n'(qb) - i_n(qa) m_n'(qa) \tag{18}$$

$$\Delta = HSkq/(H + Sp) \tag{19}$$

The temperature $T(r, \theta, t)$ can (in theory) be obtained from the inversion integral

$$T(r, \theta, t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{pt} \bar{T}(r, \theta, p) dp \tag{20}$$

provided that a real number γ can be found such that γ is greater than the real part of all the singularities of the integrand. The inversion integral will be evaluated by first finding the sum of the residues at the poles of the integrand considering $\bar{q}_n$ to be constant and then using the convolution integral, viz.,

$$\mathfrak{L}^{-1}\{\bar{q}(p)\bar{x}(p)\} = \int_0^t q(t - \tau) x(\tau) d\tau \tag{21}$$

where $\bar{x}(p) = \mathfrak{L}\{x(t)\}$ [see Eq. (7)].

The poles of the integrand of (20), occur at those values of q for which

$$q\{N_n(q, a, b) + \Delta L_n(q, b, a)\} = 0 \tag{22}$$

This is an eigenvalue equation that has the double array of eigenvalues

$$q = \alpha_{n,j} \tag{23}$$

There are no poles of the integrand of (20), with $\bar{T}$ from (15), at $p = 0$. It can be shown that for large $|q|$ and $-\pi/2 < (\arg q) \leqslant \pi/2$ the eigenvalues occur approximately at the roots of the equation

$$-H = q \tanh q(a - b) \tag{24}$$

This equation has no roots for large positive values of the real part of q, irrespective of the imaginary part of q; therefore, the contour of the inversion integral (20) can be found. There are of course many roots (a single infinity of them) of (24) for purely imaginary q. One suspects that all the roots may lie on the imaginary axis.

Although the transcendental eigenvalue equation (22) cannot be solved explicitly, a formal inversion to find $T(r, \theta, t)$ can be made. First we assume that $\bar{q}_n$ is a constant so that

$$T(r, \theta, t) = \sum_{n=0}^{\infty}\sum_{j=1}^{\infty} 2k e^{\alpha^2 kt} P_n(\cos\theta) \bar{q}_n \times \left\{\frac{L_n(\alpha, r, b) + \Delta M_n(\alpha, b, r)}{R_n(\alpha, a, b)}\right\}$$

where $\alpha = \alpha_{n,j}$ and

$$R_n(\alpha, a, b) = [(\partial/\partial q)\{N_n(q, a, b) + \Delta L_n(q, b, a)\}]_{q=\alpha} \tag{25}$$

The fully general solution when $\bar{q}_n$ is a function of p can be obtained from the convolution integral (21) to give

$$T(r, \theta, t) = \sum_{n=0}^{\infty}\sum_{j=1}^{\infty} 2k P_n(\cos\theta) \times \left\{\frac{L_n(\alpha, r, b) + \Delta_1 M_n(\alpha, b, r)}{R_n(\alpha, a, b)}\right\} \times \int_0^t \exp[\alpha^2 k(t - \tau)] q_n(\tau) d\tau \tag{26}$$

The denominator $R_n(\alpha, a, b)$ can be written in the alternative form

$$R_n(\alpha, a, b) = \left(\frac{N + \alpha^2 a^2}{\alpha^2 a}\right)\{L_n(\alpha, a, b) + \Delta_1 M_n(\alpha, b, a)\} + L_n(\alpha, b, a)\left\{\frac{HSk(3H + kS\alpha^2 + bHS\, k\alpha^2)}{(H + kS\alpha^2)^2} - \frac{N + \alpha^2 b^2}{\alpha^2 b}\right\}$$

where $\Delta_1 = \{\Delta\}_{q=\alpha}$. The value of $T_i(t)$ can now be found using (13) and (26), i.e.,

$$T_i(t) = \frac{H}{S}\int_0^t \exp\left[-\frac{H}{S}(t-\tau)\right]T(b, \theta, \tau)d\tau =$$
$$\sum_{n=0}^{\infty}\sum_{j=1}^{\infty}\frac{2kH}{S}P_n(\mu)\frac{L_n(\alpha, b, b)}{R_n(\alpha, a, b)}\int_0^t \frac{q_n(t-\tau)}{\alpha^2 k + H/S} \times (e^{\alpha^2 k\tau} - e^{-H\tau/S})d\tau \quad (27)$$

The eigenvalues of q, which are $\alpha_{nj} = \alpha = \delta + i\gamma$ (where δ and γ are real), are such that in general $\alpha^2 (= \delta^2 - \gamma^2 + 2i\delta\gamma)$ has a negative real part, and the important contributions to (26) will come from those roots for which $\delta^2 - \gamma^2$ has its largest positive value. An alternative method of obtaining a more explicit but approximate solution is by expanding $\bar{T}(r, \theta, p)$ for large values of $|p|$ and $-\pi/2 < \arg q \leqslant \pi/2$ and inverting the resulting expression term by term.[2]

The expansion for large values of $|p|$ can be obtained from the asymptotic expansion of the Bessel functions[3] of (15): Let

$$\bar{\Xi}_n(q, r) = \frac{L_n(q, r, b) + \Delta M_n(q, b, r)}{q\{N_n(q, a, b) + \Delta L_n(q, b, a)\}} = \mathfrak{L}\{\Xi_n(t, r)\}$$

then, for large $|p|$ and $-\pi < (\arg p) \leqslant \pi$,

$$\bar{\Xi}_r(q, r) \sim \frac{a}{qr}e^{q(r-a)}\left[1 + \frac{1}{q}\left\{\frac{(N+2)r - Na}{2ar}\right\} + O\left(\frac{1}{q^2}\right)\ldots + e^{-2q(r-b)}\left\{1 + \frac{1}{q}\left(-2H + \frac{Nba - 2(N+2)ar + (N+2)br}{2abr}\right)\ldots\right\} + e^{-2q(a-b)}\left\{1 + \frac{1}{q}\left(\frac{(N+2)r - Na}{2ar}\right)\ldots\right\}\right] \quad (28)$$

where $N = n(n+1)$.

The interpretation of this transform yields

$$\Xi_n(t, r) \simeq \frac{a}{r}\left[\left(\frac{k}{\pi t}\right)^{1/2}\left\{\exp\left(-\frac{(a-r)^2}{4kt}\right) + \exp\left(-\frac{(a+r-2b)^2}{4kt}\right) + \exp\left(-\frac{(3a-2b-r)^2}{4kt}\right)\right\} + \left\{\frac{k[(N+2)r - Na]}{2ar}\right\}\operatorname{erfc}\left(\frac{a-r}{2(kt)^{1/2}}\right) + \ldots + \left\{-2H + \frac{Nba + (N+2)r(b-2a)}{2abr}\right\}k \times \operatorname{erfc}\left(\frac{a+r-2b}{2(kt)^{1/2}}\right) + \ldots + \left\{\frac{(N+2)r - Na}{2ar}\right\}k \times \operatorname{erfc}\left(\frac{3a-r-2b}{2(kt)^{1/2}}\right)\ldots\right] \quad (29)$$

This series converges rapidly for all except large values of kt/r.

The fully general approximate solution (26) is then

$$T(r, \theta, t) \sim \sum_{n=0}^{\infty}\int_0^t q_n(t-\tau)\Xi(\tau, r)P_n(\cos\theta)d\tau \quad (30)$$

It will be observed that this solution, (30), valid for small values of t, does not involve H the radiation constant in the first-order terms. Thus for small values of time, the effect of this radiation at the inner surface can be ignored; a deduction in accord with the physical nature of the problem.

Prescribed Temperature at the Inner Surface

Boundary Conditions

The applied heat flux $Q_1(\theta, t)$ satisfies (1) and (2) (supra) for $t > 0$, i.e.,

$$\left[\frac{\partial T}{\partial r}\right]_{r=a} = \sum_{n=0}^{\infty} q_n(t)P_n(\cos\theta) \quad (31)$$

At the inner surface, the temperature is a prescribed function $G(\theta, t)$ of the time and θ-wise coordinate. Again we assume that it can be expanded in a series of Legendre polynomials, i.e.,

$$T(b, \theta, t) = G(\theta, t) = \sum_{n=0}^{\infty} g_n(t)P_n(\mu) \quad (32)$$

The initial condition is

$$T(r, \theta, 0) = 0 \text{ for } a \leqslant r \leqslant b \qquad 0 \leqslant \theta \leqslant 2\pi \quad (33)$$

Solution for the Temperature

The general solution (8) to the heat conduction equation (6) is still appropriate, but boundary conditions (31–33) must be applied. Thus, Eq. (11) is still valid, but the other boundary condition (32) gives

$$\bar{g}_n(p) = A_n i_n(qb) + B_n m_n(qb) \quad (34)$$

where

$$\bar{g}_n(p) = \mathfrak{L}\{g(t)\} \quad (35)$$

The operator form of the solution is

$$\bar{T}(r, \theta, p) = \sum_{n=0}^{\infty} P_n(\cos\theta) \times \left\{\frac{\bar{q}_n M_n(q, b, r) + q\bar{g}_n L_n(q; r, a)}{qL_n(q, b, a)}\right\} \quad (36)$$

in terms of the notation previously given [see (16–18)], or

$$\bar{T} = \sum_{n=0}^{\infty} P_n(\cos\theta)\{\bar{q}_n\bar{F}_n(q, r) + \bar{g}_n\bar{E}_n(q, r)\} \quad (37)$$

where

$$\bar{F}_n(qr) = \frac{M_n(q, b, r)}{qL_n(q, b, a)} \qquad \bar{E}_n(q, r) = \frac{L_n(q, r, a)}{L_n(q, b, a)} \quad (38)$$

The formal inversion of (37) proceeds as previously described in that the eigenvalues are the roots $\beta_{n,j}$ of the equation

$$L_n(q, b, a) = m_n'(qa)i_n(qb) - i_n'(qa)m_n(qb) = 0 \quad (39)$$

Again, no eigenvalue can have an unbounded real part since, for large values of $|q|$ and $-\pi/2 < \arg q \leqslant \pi/2$, the eigenvalues must approximately satisfy

$$\cosh q(a - b) = 0$$

or to second order

$$2qab \coth q(a-b) = (N+2)b - Na \quad (40)$$

The operators $\bar{F}_n(q, r)$ and $\bar{E}_n(q, r)$ are inverted separately and the convolution integral (21) (i.e., Duhamel's theorem) is used to obtain the temperature, i.e.,

$$T(r, \theta, t) = \sum_{n=0}^{\infty}\sum_{j=1}^{\infty} P_n(\cos\theta)\int_0^t \{g_n(t-\tau)E_n(\beta, r) + q_n(t-\tau)F_n(\beta, r)\}e^{\beta^2 k\tau}d\tau \quad (41)$$

where

$$E_n(\beta, r) = \frac{L_n(\beta, r, a)}{H_n(\beta, r)} \qquad F_n(\beta, r) = \frac{M_n(\beta, b, r)}{\alpha H_n(\beta, r)} \quad (42)$$

and

$$H_n(\beta, r) = \frac{1}{2k\beta}\left\{a\left(1 + \frac{N}{a^2\beta^2}\right)M_n(\beta, b, a) - bN_n(\beta, a, b)\right\} \quad (43)$$

Note that $\bar{E}_n(q, r)$ $[\bar{F}_n(q, r)]$ is not the Laplace transform of $E_n(\beta, r)[F_n(\beta, n)]$.

In these formulas (41–43), β is used to refer to a particular eigenvalue $q = \beta_{n, j}$, satisfying (39); the temperature (41) involves a sum over all these eigenvalues ($j = 1, 2 \ldots$, $n = 0, 1, \ldots$). In an earlier paper, Warren[1] gives a solution to this problem. The present method has computational advantages over Warren's solution, particularly in the neighborhood of the inner boundary $r = b$ and for small times. Indeed Warren's approach converges nonuniformly near the inner boundary $r = \bar{b}$ and takes the value zero at that boundary rather than the nonhomogeneous value of Eq. (32).

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A Steepest-Ascent Solution of Multiple-Arc Optimization Problems

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Nomenclature

$f^{(a)}$ = n-dimensional vector of known functions of x, u, and t
g = gravitational acceleration
h = altitude
m = mass
r = range
T = thrust magnitude
t = independent variable, time
t_a = corner times
u = r-dimensional vector of steering functions
v = horizontal velocity
w = vertical velocity
x = n-dimensional vector of state variable histories
β = mass flow rate
(Δt) = A-dimensional vector of corrections to the corner times
Δz = s-dimensional vector of time invariant control variable corrections
θ = thrust angle
λ = n-dimensional vector of Lagrange multipliers
φ = performance index
Ω = q-dimensional vector of constraint functions, $q \leq n$

The problem of trajectory optimization has received a great deal of attention in recent years. The problem, as usually stated, involves the determination of one or more steering functions, such as angle of attack or throttle setting, such that some performance index is optimized, subject to specified constraints. One of the most successful approaches to this problem has been the so-called steepest-ascent or gradient technique, developed by Bryson and Denham[1, 2] and Kelley.[3] The problems that have been treated with this approach, however, are actually special cases in that they involve only one subarc. This note extends the steepest-ascent technique for the simultaneous optimization of time dependent functions and time invariant quantities, originally mentioned by Denham,[2] to multiple-arc problems with the associated unknown corner times.

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The technique mentioned in Ref. 2 is applicable to the class of problems for which small variations in the constraint functions and in the performance index are related to small changes in the control variables (either time dependent or invariant) in the following manner:

$$\Delta\varphi = A_\varphi^T \Delta z + \int_{t_0}^{t_A} \lambda_{u\varphi}^T \, \delta u \, dt \tag{1}$$

$$\Delta\Omega = A_\Omega^T \Delta z + \int_{t_0}^{t_A} \lambda_{u\Omega}^T \, \delta u \, dt \tag{2}$$

In order to apply the technique of Ref. 2, the multiple-arc problem must be expressed in the form of Eqs. (1) and (2).

For general multiple-arc problems, the governing set of differential equations for the state variables can be written as follows:

$$J^{(a)} = \dot{x}^{(a)} - f^{(a)}(x, u, t) = 0 \qquad \text{for } t_{a-1} \leq t < t_a, \; a = 1, \ldots A \tag{3}$$

A complete solution to the differential equations consists of a number of segments. Each segment is referred to as a subarc, and the juncture of two subarcs is referred to as a corner. The length and position of a subarc are defined by the corner times; in particular, the ath subarc begins at t_{a-1} and ends at t_a. It is assumed that either a priori knowledge or some logical procedure is available to determine the number of subarcs in the solution.

The superscript (a) notation in Eq. (3) is used because the expressions governing the derivatives of the state variables x are not necessarily continuous from one subarc to the next. Discontinuities can result from a change in the algebraic form of the derivatives, from a discontinuity in u, or from a discontinuity in some parameter in the system.

For the present, it is assumed that t_0 and all initial conditions for the differential equations are given. The following constraints must be satisfied at the terminal point t_A:

$$\Omega = \Omega[x(t_A), t_A] = 0 \tag{4}$$

Subject to the differential constraints (3) and the terminal constraints (4) it is desired to optimize som performance index:

$$\text{performance index} = \varphi[x(t_A), t_A]$$

Because of the existence of corners, the standard relationship between the adjoint system and the linearized form of Eq. (3) is not directly applicable. An analogous result can be derived, however, which does consider the existence of corners and the associated discontinuities in the derivatives. By predicting the effects of small changes in the corner times on the terminal conditions, this analogous result permits a logical correction procedure that will ultimately yield an optimal multiple-arc solution.

Before proceeding with the derivation, one comment on notation must be made. The derivative expressions $f^{(a)}$ are defined for $t < t_a$ but are not rigorously defined for $t = t_a$. It is assumed, however, that

$$\lim_{t \to t_a} f^{(a)}(x, u, t) \quad \text{for } t_{a-1} \leq t < t_a$$

is well-behaved. Whenever it is indicated that $f^{(a)}$ is evaluated at t_a, it should be understood to mean the limit as t approaches t_a.

The derivation of the desired relationship begins with the formation of the following integral:

$$F = \sum_{a=1}^{A} \int_{t_{a-1}}^{t_a} \lambda^T J^{(a)} \, dt \tag{5}$$

Small variations in x, u, and t_a are now introduced, and only first-order terms are retained:

$$\Delta F = \sum_{a=1}^{A} \int_{t_{a-1}}^{t_a} \lambda^T \, \delta J^{(a)} dt + \sum_{a=1}^{A-1} [\lambda^T (J^{(a)} - J^{(a+1)})]_{t_a} \Delta t_a + [\lambda^T J^{(A)}]_{t_A} \Delta t_A = 0 \tag{6}$$